Discrete Mathematics
Quiz # 6 (May 28, 2015)

Name:______________________________ID:__________________________________

1. (10%) Let f be a non-decreasing function satisfying $f(n) = 8f(n/2) + 4n^3$. Please find the best big-O notation for $f(n)$.

   $\boldsymbol{O(n^3 \log n)}$

2. (20%) How many positive integers less than or equal to 500 are divisible by 5 or 7 or 11?

$$\left\lfloor\frac{500}{5}\right\rfloor + \left\lfloor\frac{500}{7}\right\rfloor + \left\lfloor\frac{500}{11}\right\rfloor - \left\lfloor\frac{500}{35}\right\rfloor - \left\lfloor\frac{500}{55}\right\rfloor - \left\lfloor\frac{500}{77}\right\rfloor + \left\lfloor\frac{500}{385}\right\rfloor$$

$$= 100 + 71 + 45 - 14 - 9 - 6 + 1 = 188$$

3. (30%) Let a_n represents the number of bit strings of length n which contain three consecutive 1s. Find $p_0, p_1, p_2,$ and p_3 for the recurrence relation $a_n = p_0 a_{n-1} + p_1 a_{n-2} + p_2 a_{n-3} + p_3 2^n$.
   Note: $p_0, p_1, p_2,$ and $p_3 > 0$

   X0 => a_{n-1} , X01 => a_{n-2} , X011 => a_{n-3} ,
   X111 => 2^{n-3}
   for $n \geq 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

   $p_0 = 1, p_1 = 1, p_2 = 1,$ and $p_3 = 1/8$ (30%)
   $a_0 = a_1 = a_2 = 0$

4. (40%) Consider the nonhomogeneous linear recurrence relation
$a_n = 3a_{n-1} - 4a_{n-3} + 9(-1)^n$. Find all solutions with $a_0 = 2, a_1 = 4, a_2 = 23$.

$\boldsymbol{x^3 - 3x^2 + 4 = 0}$
$\boldsymbol{=> (x-2)^2(x+1) = 0}$
$\boldsymbol{=> x = 2, 2, -1}$
$\boldsymbol{=> a_n{}^{(h)} = (q_0 + q_1 n)2^n + q_2(-1)^n,}$

$\boldsymbol{a_n{}^{(p)} = p_0 n(-1)^n}$
$\boldsymbol{=> p_0 n(-1)^n = 3p_0(n-1)(-1)^{n-1} - 4p_0(n-3)(-1)^{n-3} + 9(-1)^n}$
$\boldsymbol{=> p_0 n = -3p_0 n + 3p_0 + 4p_0 n - 12p_0 + 9}$
$\boldsymbol{=> 9p_0 = 9}$
$\boldsymbol{=> p_0 = 1}$
$\boldsymbol{=> a_n{}^{(p)} = n(-1)^n}$

so $\boldsymbol{a_n = a_n{}^{(h)} + a_n{}^{(p)} = (q_0 + q_1 n)2^n + q_2(-1)^n + n(-1)^n = (q_0 + q_1 n)2^n + (q_2 + n)(-1)^n}$

$\boldsymbol{2 = a_0 = q_0 + q_2}$
$\boldsymbol{4 = a_1 = (q_0 + q_1)2 - q_2 - 1}$
$\boldsymbol{23 = a_2 = (q_0 + 2q_1)4 + q_2 + 2}$

$\boldsymbol{q_0 = q_2 = 1}$
$\boldsymbol{q_1 = 2}$
$\boldsymbol{a_n = (1 + 2n)2^n + (1 + n)(-1)^n}$